



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

262. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series  $\frac{n}{(4n^2-1)^2}$ , beginning with  $n=1$ ,  $n$  being always odd.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{n}{(4n^2-1)^2} = \frac{1}{8} \left( \frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2} \right).$$

When  $n=1, 3, 5, 7, \dots$

$$\begin{aligned} \sum \frac{n}{(4n^2-1)^2} &= \frac{1}{8} \left( \frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \dots - \frac{1}{3^2} - \frac{1}{7^2} - \frac{1}{11^2} - \dots \right) \\ &= \frac{1}{8} \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right) - \frac{1}{4} \left( \frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \dots \right) \\ &= \frac{\pi^2}{64} - \frac{1}{4} (.1579 +) = \frac{\pi^2}{64} - \frac{1}{4} \cdot \frac{2\pi^2}{125} \text{ nearly, } = \frac{\pi^2}{64} - \frac{\pi^2}{250} = \frac{93\pi^2}{8000}. \end{aligned}$$

We may also write

$$\sum \frac{n}{(4n^2-1)^2} = \sum_{m=1}^{\infty} \frac{2m-1}{(16m^2-16m+3)^2} = \frac{1}{8} \int_0^1 \frac{\tan^{-1}x}{x} dx.$$

This series is discussed by William E. Heal in Vol. IX, pp. 47—49, of the MONTHLY.\*

Similar approximations were obtained by S. A. Corey, G. W. Greenwood, and J. Scheffer.

263. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the transcendentals  $e$  and  $\pi$  in the form of infinite continued fractions.

---

\*See also an article entitled "Note on the Numerical Transcendents  $S_n$  and  $s_n = S_n - 1$ ," by Professor W. Woolsey Johnson, in the current *Bulletin of the American Mathematical Society*.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

According to a method due to Euler the series

$$S = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \dots$$

may be converted into a continued fraction thus: Putting

$$S_1 = \frac{1}{B} - \frac{1}{C} + \frac{1}{D} - \frac{1}{E} + \dots \quad S_2 = \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \frac{1}{F} + \dots$$

$$S_3 = \frac{1}{D} - \frac{1}{E} + \frac{1}{F} - \frac{1}{G} + \dots \text{ etc., we get}$$

$$S = \frac{1}{A} - S_1 = \frac{1-A}{A} S_1; \therefore \frac{1}{S} = \frac{A}{1-AS_1} = A + \frac{A^2 S_1}{1-AS_1} = A + \frac{A^2}{-A + (1/S_1)}, \text{etc.}$$

$$\text{Thus, } S = \frac{1}{A + \frac{A^2}{B-A + \frac{B^2}{C-B + \frac{C^2}{D-C + \frac{D^2}{E-D + \dots}}}} \dots \dots \text{(I).}$$

Since  $\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$  we get, substituting in (I),  $A=1, B=3, C=5, D=1$ , etc.,

$$\frac{1}{4}\pi = \frac{1}{1 + \frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \dots}}}}$$

To convert the series  $\frac{1}{a} - \frac{1}{ab} + \frac{1}{abc} - \frac{1}{abcd} + \dots$  into a continued fraction, we put in (I),  $A=a, B=ab, C=abc, D=abcd$ , etc., and thus we obtain

$$\frac{1}{a} - \frac{1}{ab} + \frac{1}{abc} - \frac{1}{abcd} + \dots = \frac{1}{a + \frac{a}{b-1 + \frac{b}{c-1 + \frac{c}{d-1 + \dots}}}} \dots \text{(II).}$$

To convert  $\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} + \frac{1}{abcd} + \dots$  into a continued fraction, we have in (II) only to put  $-b, -c, -d, \dots$  for  $b, c, d, \dots$  and thus we get

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} + \frac{1}{abcd} + \dots = \frac{1}{a - \frac{a}{b+1 - \frac{b}{c+1 - \frac{c}{d+1 - \dots}}}} \dots \text{(III).}$$

Putting  $a=2, b=3, c=4, d=5, \dots$  we get

$$\frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \dots = \frac{1}{2} - \frac{2}{4} + \frac{3}{5} - \frac{4}{6} + \frac{5}{7} - \dots$$

$$\text{Hence } e=2 + \frac{1}{2} - \frac{2}{4} + \frac{3}{5} - \frac{4}{6} + \frac{5}{7} - \dots$$

Also solved by G. W. Greenwood, and G. B. M. Zerr.

264. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the invariant  $2(a_0a_4 - 4a_1a_3 + 3a_2^2)$  of the binary quartic  $a_0x_1^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4$  in terms of roots of the latter.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

It can be shown that, if

$$\begin{aligned} & a_0x_1^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 - 4a_3x_1x_2^3 + a_4x_2^4 \\ & \equiv a_0(x_1^2 + 2px_1x_2 + qx_2^2)(x_1^2 + 2p'x_1x_2 + q'x_2^2), \end{aligned}$$

then  $4\theta^3 - I\theta + J = 0$ , where  $I = a_0a_4 - 4a_1a_3 + 3a_2^2$ , and  $\theta = a_2 - a_0pp'$ .

Let  $\beta, \gamma$  be the roots of  $x_1^2 + 2px_1x_2 + qx_2^2 = 0$ , and  $a, \delta$  the roots of  $x_1^2 + 2p'x_1x_2 + q'x_2^2 = 0$ . Then

$$\theta = \frac{a_0}{6} \Sigma \beta\gamma - \frac{a_0}{4}(\beta + \gamma)(a + \delta) = \frac{a_0}{12}(v - w),$$

where  $u = (\beta - \gamma)(a - \delta)$ ,  $v = (\gamma - a)(\beta - \delta)$ ,  $w = (a - \beta)(\gamma - \delta)$ . The roots of the reduced cubic are therefore,

$$\frac{a_0}{12}(u - v), \frac{a_0}{12}(v - w), \frac{a_0}{12}(w - u).$$

It is easily found that  $u + v + w = 0$ . Consequently,  $\Sigma vw = -\frac{1}{2} \Sigma u^2$ , and

$$\frac{I}{4} = - \Sigma \theta_1 \theta_2 = - \frac{a_0^2}{144} \Sigma (uv - uw + vw - v^2) = \frac{a_0^3}{144} \cdot \frac{3}{2} \Sigma u^2.$$

Hence  $I = \frac{a_0^2}{24} \Sigma u^2$ , where  $u, v, w$  have the values given above.

Also solved by G. B. M. Zerr.